TRINITY COLLEGE FOR WOMEN
NAMAKKAL
Department of Mathematics

PARTIAL DIFFERENTIAL EQUATIONS (Part-II)

19PMA08-ODD Semester

Presented by
Dr.R.MALARVIZHI
Assistant Professor & HOD-PG,
Department of Mathematics
http://www.trinitycollegenkl.edu.in/

Explain canonical forms of Elliptic, Hyperbolic, and Parbolic :

Reduce the PDE

$$R_r + S_s + T_t + f(x,y,z,p,q) = 0 \quad \text{---------} \ (1)$$

to a canonical form we apply the transformation.

$$\xi = \xi(x,y) \quad , \quad \eta = \eta(x,y) \quad \text{--------} \ (2)$$

such that, the function ξ and η are continuously differentiable and the jacobian

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x\eta_y - \eta_x\xi_y \neq 0 \quad \text{--------} \ (3)$$

in the domain 'Ω' , where equation (1) holds, Now we have,

$$P = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$P = Z_\xi \, . \, \xi_x + Z_\eta \, . \, \eta_x$$

$$p = \xi x \, . \, Z\xi + \eta_x + Z_\eta$$

$$q = \xi_y \, . \, Z_\xi + \eta_y + Z\eta$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$r = \frac{\partial}{\partial x}\left(\xi_x \, . \, Z_\xi + \eta_x \, . \, Z_\eta\right)$$

$$r = \frac{\partial}{\partial \xi}\left[\xi_x \, . \, Z_\xi + \eta_x \, . \, Z_\eta\right]\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left[\xi_x Z_\xi + \eta_x Z_\eta\right]\frac{\partial \eta}{\partial x}$$

$$r = \xi_x^2 Z_{\xi\xi} + \xi_{xx} Z_\xi + 2 \xi_x \eta_x Z_{\eta\xi} + \eta_{xx} Z_\eta + \eta_x^2 Z_{\eta\eta}$$

Simlarly ,

$$t = \xi_y^2 Z_{\xi\xi} + \xi_{yy} Z_\xi + 2 \xi_y \eta_y Z_{\eta\xi} + \eta_{yy} Z_\eta + \eta_y^2 Z_{\eta\eta}$$

$$s = \xi_x \xi_x Z_{\xi\xi} + \eta_x \eta_y Z_{\eta\eta} + \xi_x \eta_y Z_{\eta\xi} + \xi_y \eta_x Z_{\eta\xi} + \xi_x \eta_y Z_\xi + \eta_x \eta_y Z_\eta$$

Substitute these values of p,q,r,s,t in (1) we get

$$A(\xi_x, \xi_y) Z_{\xi\xi} + 2B(\xi_x, \xi_x, \eta_x, \eta_y) Z_{\eta\xi} + A(\eta_x, \eta_y) Z_{\eta\eta}$$

$$+ F(\xi, \eta, Z, Z_\xi, Z_\eta) = 0 \quad \text{-------------} \quad (4)$$

Where,

$$A(U,V) = Ru^2 + Suv + Tv^2$$

$2B(u_1, u_2, v_1, v_2) = 2Ru_1u_2 + S(u_1v_2 + u_2v_1) + 2Tv_1v_2$

Now it can be easily verified that,

$2B^2(\xi_x, \xi_y, \eta_x, \eta_y) - A(\xi_x, \xi_y) A(\eta_x, \eta_y) = (S^2 - 4RT) J$ ----- (5)

where J is given by (3)

Case (i): $S^2 - 4RT > 0$

Under the condition $S^2 - 4RT > 0$, the equation $R\lambda^2+S\lambda+T=0$ has real and distinct roots.

Let these roots be λ_1 and λ_2

We choose ξ and η such that $\xi_x = \lambda_1\xi_y$, $\eta_x = \lambda_2\eta_y$

Now,

$\xi_x = \lambda_1\xi_y$ --------- (6)

$\xi x - \lambda_1\xi_y = 0$ being a first order lineat PDE.

we've, $\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{d\xi}{0}$

Therefore, $d\xi = 0$ $\quad$ ξ = constant

$$\frac{dx}{1} = \frac{dy}{-\lambda_1}$$

$$\frac{dx}{1} = \frac{dy}{-\lambda_1}$$

$$\frac{dy}{dx} + \lambda_1(x,y) = 0 \, , \, \frac{dy}{dx} + \lambda_2(x,y) = 0 \qquad \text{--------} \ (7)$$

Let the solution of these equations be given by,

$$f_1(x,y) = \text{constant}$$

$$f_2(x,y) = \text{constant}$$

Thus we get,

$$\xi = f_1\,(x,y) \quad \text{and} \quad \eta = f_2(x,y) \qquad \text{---------} \ (8)$$

Now,

$$A(\xi_x\,,\,\xi_y) = R\xi_x^{\,2} + S\xi_x\xi_y + T\xi_y^{\,2}$$

$$= R\lambda_1^{\,2}\xi_y^{\,2} + S\lambda_1\xi_y\xi_y + T\xi_y^{\,2}$$

$$= \xi_y^{\,2}\,(0) \quad [\text{ since, } \lambda_1 \text{ is a root of } R\lambda^2+S\lambda+T=0]$$

$$A(\xi_x\,,\,\xi_y) = 0$$

Similarly,

$A(\eta_x,\eta_y) = 0$ [since, λ_2 is a root of $R\lambda^2+S\lambda+T = 0$]

$B^2 = (S^2 - 4RT)\, J \neq 0$

Equation (4) reduces to

$$Z_{\xi\eta} = g\,(\xi,\eta,Z,Z_\xi,Z_\eta)$$

which is required canonical form for the hyperbolic PDE.

Case (ii) :

Then $R\lambda^2$ + $S^2 - 4RT = 0$ equal roots.

i.e) $\lambda_1 = \lambda_2 = \lambda$ (say)

we choose $\xi = f_1(x,y)$, $f_1(x,y)$ = constant is a solution of $\frac{dy}{dx} + \lambda(x,y) = 0$

where $\dfrac{A\,(\xi_x,\xi_y) = 0}{S^2 - 4RT = 0}$

Therefore equation (5) implies, $B = 0$

However,

$$A(\eta_x,\eta_y) \neq 0$$

Otherwise η will depend upon 'ξ' using $A = B = 0$ in (4),we get

$$Z_{\eta\eta} = g(\xi,\eta,Z,Z_\xi,Z_\eta)$$

which is the required canonical form for the parabolic PDE

Case(iii) :

$S^2 - 4RT < 0$

In this case the roots of $R\lambda^2+S\lambda+T=0$ are imaginary.

Therefore , ξ and η will be complex.

Let $\xi = \alpha + i\beta$

$\eta = \alpha - i\beta$

where, α , β are equal

$$\alpha = 1/2\ (\xi + \eta)$$

$$\beta = 1/2(\eta - \xi)$$

with this transformation we have

$$Z_{\xi\eta} = \frac{1}{4}\ (Z_{\alpha\alpha} + Z_{\beta\beta})$$

and proceeding on the similar lines as on case (i) we get,

$$Z_{\alpha\alpha} + Z_{\beta\beta} = \Phi\ (\alpha,\beta,Z,Z_{\alpha} ,Z_{\beta})$$

which is the required canonical form for the Elliptical PDE.

Conclusion :

i) **If $S^2 - 4RT < 0$, then it is Elliptic.**

ii) **If $S^2 - 4RT = 0$, then it is Parabolic.**

iii) **If $S^2 - 4RT > 0$, then it is Hyperbolic.**

THANK YOU

http://www.trinitycollegenkl.edu.in/