TRINITY COLLEGE FOR WOMEN
NAMAKKAL
Department of Mathematics

REAL ANALYSIS II
21PMA06-Even Semester

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THE RIEMANN-STIELTJES INTEGRAL

DEFINITIONS:

- **Refinement:**

The partition P^* is a refinement of P if $P^* \supset P$ .Given two partitions, P_1 and P_2 , we say that $P^*$is their common refinement if $P^*=P_1 \cup P_2$.

- **Partition:**

Let [a,b] be a given integral. By a partition of P of [a,b], we mean a finite set of points $(x_0, x_1, \ldots, x_n)$, where

$$a=x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n=b$$

ie) $P= \{a=x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n=b\}$

➢ Upper And Lower Integrals Corresponding to of f Over [a,b]:

Let α be a monotonically increasing functions on [a,b] (since $\alpha(a)$ and $\alpha(b)$ are finite , it follows that α is bounded on [a,b]).

Corresponding to each partition P of [a,b],we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

It is clear that $\Delta\alpha_i = 0$

NOTE:

Since f is bounded, then there exixts two numbers, m and M such that

$$m \le f(x) \le M \qquad (a \le x \le b)$$

Hence for every P, we have

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

So that the numbers L(P,f) and U(P,f) form a bounded set. This shows that the upper and lower integrals are defined for every bounded function f.

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SOME IMPORTANT RESULTS:

(a) If P* is a refinement of P, Then

(i) $L(P,f,\alpha) \leq L(P^*,f,\alpha)$

(ii) $U(P^*,f,\alpha) \leq U(P,f,\alpha)$

(b) If f is continuous on [a,b], then $f \in \Re(\alpha)$ on [a,b].

(c) Suppose f is bounded on [a,b], f has only finitely many points of discontinuity on [a,b] and α is continuous at every point at which f is discontinuous. Then $f \in \Re(\alpha)$.

PROPERTIES OF THE INTEGRAL :

1. If $f_1 \in \Re(\alpha)$ and $f_2 \in \Re(\alpha)$ on [a,b]

Then:

(i) $f_1 + f_2 \in \Re(\alpha)$

(ii) $Cf \in \Re(\alpha)$ for every constsnts C

(iii) $\int_a^b (f_1 + f_2) = \int_a^b f_1 + \int_a^b f_2$

(iv) $\int_a^b c\, f\, d\alpha = c\int_a^b f\, d\alpha$

2. If $f_1(x) \le f_2(x)$ on [a,b], Then

$$\int_a^b f_1 \, d\alpha \le \int_a^b f_2 \, d\alpha$$

3. If $f \in \Re(\alpha)$ on [a,b] and if $a<b<c$, Then $f \in \Re(\alpha)$ on [a,b] and on [c,b] and

$$\int_a^c f \, d\alpha + \int_c^b f \, d\alpha = \int_a^b f \, d\alpha$$

4. If $f \in \Re(\alpha)$ on [a,b] and if $|f(x)| \leq M$ on [a,b], Then

$$\left| \int_a^b f \, d\alpha \right| \leq M [\alpha(b) - \alpha(a)]$$

5. If $f \in \Re(\alpha_1)$ and $f \in \Re(\alpha_2)$, Then $f \in \Re(\alpha_1 + \alpha_2)$

$$\int_a^b f \, d(\alpha_1 + \alpha_2) = \int_a^b f \, d(\alpha_1) + \int_a^b f \, d(\alpha_2)$$

If $f \in \Re(\alpha)$ and c is a positive constants,Then $f \in \Re(\alpha)$ and $\int_a^b f \, d(c\alpha) = c \int_a^b f \, d\alpha$

INTEGRATION OF VECTORS – VALUED FUNCTIONS :

Let $f_1, f_2, \ldots, f_k$ be real functions on [a,b] and let $f = (f_1, f_2, \ldots, f_k)$ be the corresponding mapping of [a,b] into R^k . If α increases monotonically on [a,b], to say that $f \in \Re(\alpha)$ means that $f_j \in \Re(\alpha)$ for j= 1,2,...,k. If this is the case, we define

$$\int_a^b f\, d\alpha = \left(\int_a^b f_1\, d\alpha, \ldots, \int_a^b f_k\, d\alpha \right)$$

THANK YOU

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